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CO-POLAR AND CO-AXIAL TRIANGLES IN CONICS.

By REV. ALAN S. HAWKESWORTH, Pittsburg, Penn.

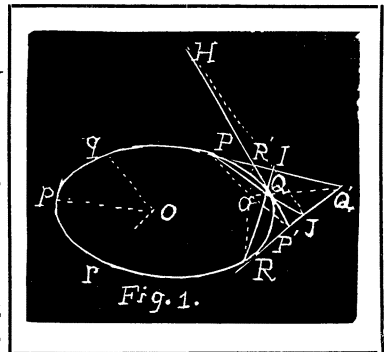
Theorem 1. If an inscribed and a circumscribed triangle to any conic curve touch in the same three points, then will they be co-polar in a point within the curve; and co-axial in the polar of said point [Figs. 1 and 2].

Draw the tangents $PR'Q'$, $R'QP'$, and $RP'Q'$, of any three points PQR upon any conic curve, meeting each other, and forming the circumscribing triangle $P'Q'R'$. And join PQR , forming an inscribed triangle, which touches the conic in the same three points as $P'Q'R'$. Draw Cp , Cq , and Cr , the semi-diameters in ellipse [Fig. 1], hyperbola, or circle, parallel to the tangents at P , Q , and R , respectively. Or in the parabola, the lines from any point C to the axis, or any diameter, parallel to said tangents.

Then, by a well known theorem, $PR' = R'Q = Cp : Cq$, and $RQ' = PQ' = Cr : Cp$, and $QP' = P'R = Cq : Cr$. So that, multiplying and cancelling, $PR' \cdot RQ' \cdot QP' = R'Q \cdot PQ' \cdot P'R$. And therefore, by a known theorem in "Transversals," lines PP' , QQ' , and RR' must concur in a point O , or the inscribed and circumscribed triangles PQR , and $P'Q'R'$ be co-polar in O .

But as so co-polar, they must also be co-axial, by a known theorem, a fact also evident from "Poles and Polars." For if chords PQ and RQ cut the tangent of R and P in J and I , respectively, while line JI and chord PR concur in H , and HQ be joined, then will the three summits of quadrangle $PIJR$ be H , Q' , and Q on the curve. So that $Q'Q$ is the polar of H ; and HQ thus the tangent at Q , identical with $R'Q'P'$.

Next, HIJ , the co-axial line of triangles PQR and $P'Q'R'$, is also the polar, in respect to the conic, of O , their co-polar point. For $RP'JQ'$ being a harmonic range of quadrangle $PIJR$; PR , POP' , PQJ and PIQ' is also a harmonic pencil; and POP' thus the polar of I , and its chord of tangential contact. Similarly, $PR'IQ'$ being a harmonic range, gives us

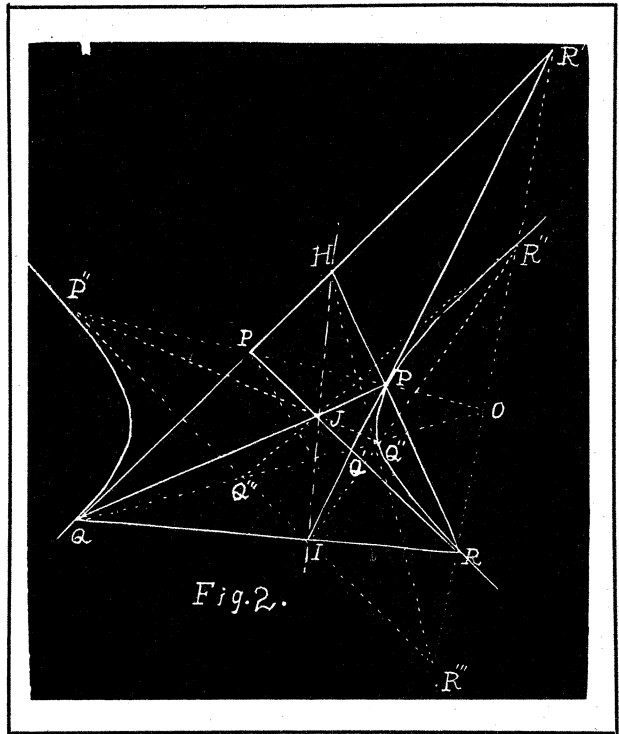


RP , ROR' , RQI and RJQ' as a harmonic pencil; and ROR' thus as the polar of J , while $Q'QO$ is the polar of H . So that O , the common point of POP' , ROR' and $Q'QO$, must be the pole to polar HIJ , in respect to the conic.

And lastly, the co-polar point O ever falls within the curve. For when an ellipse, circle, parabola, or one branch of a hyperbola is in question, then the co-polar point O must obviously fall within the inscribed triangle PQR , and hence also within the curve, while, when points PQ , say, lie on one branch, and R on the opposite branch of a hyperbola [Fig. 2], then the co-axial line HIJ plainly cuts internally, the two triangles PQR and $P'Q'R'$, and hence must lie between the branches, so that its pole O still lies within the curve, and within that branch upon which lies the base PR of the inscribed triangle, but now beyond said base and between the produced sides PQ and RQ .

Theorem 2. If there be an inscribed and a circumscribed triangle to any conic curve, touching it in the same three points, and thus co-polar in a point O , within the curve, and co-axial along the polar of said point, and if the three radiant co-polar points cut the conic again in three fresh points and these points be joined, and their tangents be also drawn, then the resultant four triangles—two inscribed, and two circumscribed—will be commonly co-polar in the radiant point O within the curve; and also will be, not merely commonly co-axial along its polar, but also co-axial in the same three fixed points HIJ .

Take any conic—say both branches of the hyperbola [Fig. 2]—and let PQR and $P'Q'R'$ be an inscribed and circumscribed triangle, touching it in the same three points P , Q , and R , and hence co-polar in O within the curve, and co-axial in HIJ along its polar [Theorem 1]. Let the three radiant lines $OPP'P'$, $OQ'Q'Q$, and $ROR''R'$ cut the conic again in points P' , Q' , and R'' , respectively. Join these three fresh points, and draw $P'Q''R''$, $Q'P''R''$, and $R'P''Q''$, their tangents, cutting in Q'' , R'' , and



P''' , respectively, giving us, therefore, two inscribed triangles, PQR and $P''Q''R''$, and two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$. We will now show that O is the common co-polar point of all four triangles, and HJI , along its polar, their three common collinear co-axial points.

For, by construction, O is the co-polar point of the inscribed triangles PQR and $P''Q''R''$. And if their sides PR and $P''R''$ cut in, say h , while QR and $Q''R''$ cut in, say i , and PQ , $P''Q''$ cut in, say j ; then will O and h be summits of quadrangle $PRR''P''$, and likewise Oi summits of quadrangle $QRR''Q''$, and Oj summits of quadrangle $PQQ''P''$. Hence hji is the polar of O with respect to the conic; thus coinciding with line HJI . But the three points where the polar of O , the co-axial line of triangles PQR and $P'Q'R'$, cuts the sides of triangle PQR , are HJI . So that lines HJI and hji not only coincide; but points Hh , Jj , and Ii , respectively, are identical; so that HJI are the common collinear co-axial points of the three co-polar triangles PQR , $P'Q'R'$, and $P''Q''R''$; their corresponding sides PR , $P'R'$, and $P''R''$ concurring in H ; sides PQ , $P'Q'$, and $P''Q''$ concurring in J , and QR , $Q'R'$, and $Q''R''$ in I .

Then lastly, H being a summit of the quadrangle $PJIR$, must be the pole to the polar $QQ'Q''O$ joining its other two summits Q' and Q , and hence in it meet the two tangents $QP'HR'$ and $R'''Q''P'''H$, while the quadrangle $PJIR$ again gives us $RQ'JP'$ as a harmonic range. And thus $PQ'I$ and $OPP'P''$ as conjugate rays to the pencil $PQ'I$, PJQ , $PP'P''$, and RPH . So that in I concur the tangents $R'PQ'I$ and $P''Q'''IR'''$. While similarly, $RQ'JP'$ and $RR''R'$ being conjugate rays in the pencil RQ , RP' , RH , and RR' , it follows that in J concur the tangents $RQ'J$, and $R''P'''JQ'''$.

So that HJI are the three collinear points in which the two inscribed triangles PQR and $P''Q''R''$, and the two circumscribed triangles $P'Q'R'$ and $P'''Q'''R'''$ are commonly co-axial, while by Theorem 1, triangles $P''Q''R''$ and $P'''Q'''R'''$, being so co-axial in HJI , must be co-polar in O , its pole, and thus all four triangles be commonly co-polar in said point O . So that $OPP'''P'P''$ are collinear; as also $OQ''Q'Q'''Q$, and $R''ROR'R'$.

NOTE ON THE QUARTIC.

By DR. R. P. STEPHENS, Wesleyan University, Middletown, Conn.

The general quartic equation

$$(1) \quad ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

where the coefficients may be real or complex, can be reduced to the form

$$(2) \quad x^4 - 2ux^2 + v^2 = 0.$$